

CAUCHY PROBLEM FOR NLKG IN MODULATION SPACES WITH NONINTEGER POWERS

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ABSTRACT. In this paper, we consider the Cauchy problem for the nonlinear Klein-Gordon equation whose nonlinearity is $|u|^k u$ in the modulation space, where k is not an integer. Our method can be applied to other equations whose nonlinear parts have regularity estimates. We also study the global solution with small initial value for the Klein-Gordon-Hartree equation. By this we can show some advantages of modulation spaces both in high and low regularity cases.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the Cauchy problem for the following nonlinear Klein-Gordon equation (NLKG):

$$u_{tt} + (I - \Delta)u = \pm |u|^k u, \quad u(0) = u_0, u_t(0) = u_1, \quad (1.1)$$

where

$$k \in (0, +\infty) \setminus \mathbb{Z}^+, u_{tt} = \partial^2 / \partial^2 t,$$

and

$$\Delta = \partial^2 / \partial^2 x_1 + \dots + \partial^2 / \partial^2 x_n$$

is the Laplace operator. It is well known that the NLKG has the following equivalent integral form:

$$u(t) = K'(t)u_0 + K(t)u_1 - \int_0^t K(t - \tau) |u|^k u d\tau, \quad (1.2)$$

where we denote $\omega = (I - \Delta)$ and

$$K(t) = \frac{\sin t\omega^{\frac{1}{2}}}{\omega^{\frac{1}{2}}}, \quad K'(t) = \cos t\omega^{\frac{1}{2}}.$$

The aim of this paper is to study the local and global well posedness of NLKG in modulation spaces. The modulation space $M_{p,q}^s$ was originally introduced by Feichtinger in [4], where its definition is based on the short-time Fourier transform and the window function. Feichtinger's initial motivation was to use the modulation space to measure smoothness for some function or distribution spaces. Since then, this space was received an extensive study on its analysis/topological construction and algebraic properties. See, for example, [6, 10, 11, 13] and the references therein for more details. Later, people found that this space is a good working

Date: November 11, 2014.

2000 Mathematics Subject Classification. 35A01, 35A02, 42B37.

Key words and phrases. Modulation spaces, Nonlinear Klein-Gordon equation, Cauchy problem, noninteger power.

frame to study certain Cauchy problems of nonlinear partial differential equations. To this end, Wang and Hudzik gave another equivalent definition in [22] using the frequency-uniform-decomposition operators. With this discrete definition, they was able to consider the global solutions for nonlinear Schrödinger equation and nonlinear Klein-Gordon equation in the space $M_{p,q}^s$. Following their pioneer work, we can find a lot of research papers in the literature that address various harmonic analysis and PDE problems on the modulation spaces. In the following, we list a few of these results, among numerous of papers. Gröbner in his PH.D. thesis [7] introduced the α modulation spaces that reveals the essential connection between the modulation spaces and the Besov spaces; Han and Wang[8] followed Gröbner's idea to give a discrete version of the α modulation space based on the frequency-uniform-decomposition, and they obtained more properties related to this space; Feichtinger, Huang and Wang [5] studied the trace operator in modulation, α -modulation and Besov spaces. Also, for PDE problems, Wang, Zhao, and Guo [23] studied the local solution for nonlinear Schrödinger equation and Navier-Stokes equations; Wang and Huang [21] obtained the local and global solutions for generalized KdV equations, Benjamin-Ono and Schrödinger equations; Tsukasa Iwabuchi studied the local and global solutions for Navier-Stokes equations, as well as the heat equations (see [9]). However, we observe that the nonlinear parts of above mentioned equations is either $|u|^k u$ with $k \in \mathbb{Z}^+$ or a multi-linear function $F(u, \dots, u)$. The reason of such a restriction is that all the estimates in [23], [21], [9] are based on the algebra property

$$\|u^2\|_{M_{p,1}^s} \preceq \|u\|_{M_{p,1}^s}^2,$$

that causes that the exponent k must be a positive integer in the nonlinear term $|u|^k u$. In a recent article [16], Ruzhansky, Sugimoto and Wang stated some new progresses on the modulation spaces. In the same article they posed three open questions. One of these questions is to study nonlinear PDE whose nonlinear term $|u|^k u$ has a non-integer $k \in (0, +\infty)$.

Motivated by the above question, in this paper, we will give a partial answer to the above problem. Before we attack the problem, let us briefly describe some obvious difficulties in handling this problem. Unlike the Lebesgue spaces or the Besov spaces, we do not have the Littlewood-Paley theory or any of its analog on the modulation spaces. So we can only use its algebra property and local analysis on its windows in the frequency spaces. However, both of these two tools handle only the case of integer k . These bring the main difficulty in our problem. To achieve our aim, fortunately we observe that modulation spaces and Besov spaces can be embedded each other. So, we can use this property to deduce the problem on the modulation spaces to those on the Besov spaces and then transfer the obtained estimates back to the modulation spaces. Of course, during this embedding process, we might loss some regularity. So this method might be applied in some equations whose nonlinear parts have regularity estimates. These equations include the Klein-Gordon equation, the heat equation and some other equations with regularity estimates.

Now we state our main theorems in the following.

Theorem 1. *Let $s \in \mathbb{R}, k \geq [s], 2 < p < \infty$, and $1 \leq q < \infty$. Assume that q satisfies the following conditions:*

$$\max\{1 + \frac{n}{2} - \frac{n}{q}, \frac{n}{2} - \frac{1}{k}[1 - (\frac{1}{q} - \frac{1}{2})n]\} < s < \frac{n}{2} \quad (1.3)$$

when $q < 2$ and $n(\frac{1}{q} - \frac{1}{2}) < 1$;

and

$$\max\{1, \frac{n}{q'} - \frac{1}{k}[1 - n(\frac{1}{2} - \frac{1}{q})]\} < s < \frac{n}{q'} \quad (1.4)$$

when $q > 2, n(\frac{1}{2} - \frac{1}{q}) < 1$.

For any $(u_0, u_1) \in M_{2,q}^s \times M_{2,q}^{s-1}$, there exists a $T > 0$ such that the equation (1.1) has a unique solution in the space

$$L^\rho(0, T; M_{p,q}^{s-\beta}) \cap L^\infty(0, T; M_{2,q}^s), \quad (1.5)$$

where ρ and β are any real numbers satisfying $\rho\beta = \frac{n+1}{n-1}$ and $\beta \in (0, \frac{n+1}{2n-2})$.

Since $\beta > 0$ is arbitrary, $L^\rho(0, T; M_{p,q}^{s-\beta})$ might be closed to the space $L^\infty(0, T; M_{p,q}^s)$, which has more regularity and better integrality.

Remark 1. When consider local well-posedness of NLKG in Besov space $B_{p,q}^s$, the domain of p, q is $1 \leq q \leq \infty$ and $(\frac{1}{2} - \frac{1}{p})n \in [0, 1)$. Comparing this to Theorem 1, we can see that the domain of p in the modulation space is similar to the domain of q in the Besov space, and q behaves in modulation spaces quite like the performance of p in the Besov space. One can easily understand this nature, when one compares the embedding relations between the modulation spaces (see 2.1) to the Sobolev embedding in Besov spaces.

In Theorem 1, we use an auxiliary space $L^\infty(0, T; M_{2,q}^s)$. The purpose is that we want to expand the domain of p, q by the Strichartz estimate. Actually, we can also prove the unconditional local well posedness in $L^\gamma(0, T; M_{p,q}^s)$ without the auxiliary space. As we all know, the unimodular semigroup $e^{it|\Delta|^{\alpha/2}}$ is bounded on the L^p space or on the Besov space $B_{p,q}^s$ if and only if $p = 2$ (or $\alpha = 1$ at $n = 1$). So if we want to obtain the local well posedness in $B_{p,q}^s$ when $p \neq 2$, we should use the Sobolev space H^s as an auxiliary space. On the other hand, for the Sobolev space H^s alone, it is difficult to estimate nonlinear part when $s < \frac{n}{2}$, so it also needs the Besov space as an auxiliary space. But in modulation spaces, we are able to obtain the $M_{p,q}^s$ -boundedness for the unimodular semigroup and we can also estimate the nonlinear part in the low regularity case. As these advantages, the next corollary shows that we can obtain the following unconditional local well posedness in modulation spaces. This feature is not available either on the Besov spaces, or on the Sobolev spaces, when one studies the problem in the low regularity case.

Corollary 1. *Let $1 \leq q < \infty, 2 \leq p < \infty, s \in \mathbb{R}$ and $k \geq [s]$. Assume that they satisfy $q \in [p', p], (1 - \frac{2}{p})n < 1$ and*

$$\max\{1 - n(\frac{1}{q} - \frac{1}{p}), \frac{n}{q'} - \frac{1}{k}\} < s < \frac{n}{q'} - \frac{1}{k}(1 - \frac{2}{p})n. \quad (1.6)$$

Then for any initial value $(u_0, u_1) \in M_{p,q}^s \times M_{p,q}^{s-1}$, there exists a $T > 0$ such that the equation (1.1) has an unique solution in $L^\gamma(0, T; M_{p,q}^s)$ for any $\gamma \geq k + 1$.

Remark 2. Throughout the proof, we will use the Strichartz estimate and a nonlinear estimate in Besov spaces. Both of these two tools requires $2 < p < \infty$. So, for the case $1 \leq p \leq 2$, we are not able to obtain the well-posedness in the Besov spaces. However, such a restriction can be removed when we study the same problem on the modulation spaces. In the next theorem, we indeed obtain the solution in $M_{p,q}^s$ when $1 \leq p \leq 2$. Recall that in the Besov spaces $B_{p,q}^s$, if we want to use Sobolev embedding to control the norm $B_{p,q}^s$ by the norm $B_{p_1,q}^{s_1}$ with $p_1 < p$, it needs more regularity $s_1 > s$. But in modulation spaces $M_{p,q}^s$, we have uniformly estimate for the index p (see (2.1)) which has no influence to the regularity index s . Hence we can use this embedding property and boundedness of unimodular semigroup to solve the problem in the case $1 \leq p \leq 2$. Specifically, we have the following theorem:

Theorem 2. Let $1 < p \leq 2$, $1 \leq q < \infty$, $s \in \mathbb{R}$ and $k \geq [s]$. Assume that they satisfy that $q \in [p, p']$, $(\frac{2}{p} - 1)n < 1$, and

$$\max\{1 - n(\frac{1}{q} - \frac{1}{p'}), \frac{n}{q'} - \frac{1}{k}\} < s < \frac{n}{q'} - \frac{1}{k}(1 - \frac{2}{p'})n. \quad (1.7)$$

Then the same conclusion as Corollary 1 holds.

Remark 3. In the case $p = 2$, the condition $q \in [p', p]$ in the above theorem means $q = 2$. In [22], Wang and Hudzik proved that the space $M_{2,2}^s$ is equivalent to the Sobolev space H^s , so this result is not interesting. Actually, when $p = 2$, it is not necessary to choose $q = 2$. The range of q for $p = 2$ can be wider. We will address this special case in Remark 6.

Now we turn to state the global solution of NLKG in modulation spaces.

Theorem 3. Let $1 \leq q < \infty$, $2 \leq p < \infty$, $s \in \mathbb{R}$ and $k \geq [s]$. Assume that they satisfy $q \in [p', p]$, $(1 - \frac{2}{p})n < 1 - \alpha$ and

$$\max\{1 - \frac{\alpha}{2} - n(\frac{1}{q} - \frac{1}{p}), \frac{n}{q'} - \frac{1}{k}(1 - \alpha) + \frac{\alpha}{2}\} < s < \frac{n}{q'} - \frac{1}{k}(1 - \frac{2}{p})n + \frac{\alpha}{2} \quad (1.8)$$

where

$$\alpha = \theta(n+1)(\frac{1}{2} - \frac{1}{p}), \quad \delta = \theta(n-1)(\frac{1}{2} - \frac{1}{p}). \quad (1.9)$$

for some $\theta \in [0, 1]$. In addition, assume $q \in [\gamma', \gamma]$, $\gamma \geq \frac{2}{\delta}$, and $k > \frac{4}{\theta} + \frac{2}{n}$. Then there exists a small $\nu > 0$ such that for any $\|u_0\|_{M_{p,q}^s} + \|u_1\|_{M_{p,q}^{s-1}} \leq \nu$, equation (1.1) has an unique global solution

$$u \in L^\infty(R; M_{p,q}^s) \cap L^{k+2}(R; M_{p,q}^{s-\frac{\alpha}{2}}). \quad (1.10)$$

Remark 4. When we choose $\theta = 1$ in Theorem 3, we can find the global solution with small initial value for $k > 4 + \frac{2}{n}$. In [22], Wang and Hudzik proved that if k is an integer, the global existence interval is $k \geq \frac{4}{n}$ which is wider than ours. That is because during the embedding between Modulation spaces and Besov spaces we lost some regularity. Hence we need more power of u to guarantee more regularity to make up this lost.

By far, we find the local solution for $1 < p < \infty$ and the global solution for $2 < p < \infty$ when k is not integer. In all cases, nonlinear estimate in modulation spaces relies heavily on the corresponding nonlinear estimate in Besov spaces and some regularity is lost. However, we obtain two advantages in the modulation spaces

and they are quite unique comparing to the results on the Besov space. First, in Corollary 1, we obtain the unconditional local well posedness with low regularity. Second, in Theorem 2, we solve the problem in the case $1 < p < 2$. Moreover, if the nonlinearity is a multi-linear function, we can find that the modulation spaces have many other advantages. Below, we will use the nonlinear Klein-Gordon-Hartree equation(NLKGH):

$$u_{tt} + (I - \Delta)u + (|x|^{-\mu} * u^2)u = 0, u(0) = u_0, u_t(0) = u_1 \quad (1.11)$$

to illustrate these advantages.

We consider the global solution with small initial value for equation (1.11) in $\mathbb{R} \times \mathbb{R}^3$, and compare the result with the same solution in Besov spaces obtained in [12]. As we mentioned before, the role of the index q in the modulation space is significant. The regularity index s might depend on q in some estimates, for instance see (2.2). In order to obtain a good time-space estimate, in [22] Wang and Hudzik gave up a traditional method of dual estimate. They introduced the space $l_{\square}^{s,q}(L^r(0, T; L^p(\mathbb{R}^n)))$ (see Definition 1) to replace the standard modulation space. Then, for $q \in [\gamma', \gamma]$, they were able to invoke the Minkowski inequality to obtain some time-space estimates in $L^r(0, T; M_{p,q}^s)$. Look back to Theorem 3, in the proof we need the embedding between modulation spaces and Besov spaces to solve the case $k \notin \mathbb{Z}$, so we work on the space $L^r(0, T; M_{p,q}^s)$ whose Stricharz estimate needs to restrict to $q \in [\gamma', \gamma]$. But for the equation (1.11), we want to use the relation between s and q to extend the domain of μ . So we hope that the restrictions on q are as less as impossible. To achieve this target, we choose the space $l_{\square}^{s,q}(L^r(R; L^p(\mathbb{R}^n)))$ to find the global solution and establish the following result.

Theorem 4. *Suppose that $(u_0, u_1) \in M_{2,q}^s \times M_{2,q}^{s-1}$, where $1 < q < \infty, s \geq 0, \alpha$ and δ are defined in (1.9). Assume that the domain of p satisfies*

$$(1 - \frac{2}{p}) \in [\frac{1}{2\theta(n-1)}, \frac{1}{\theta(n-1)}).$$

For $2n(1 - \frac{2}{p}) \leq \mu \leq 2(s + \frac{n}{q}) + 1 - 2\alpha - n$, we can find a constant $\varepsilon > 0$ for which if

$$\|u_0\|_{M_{2,q}^s} + \|u_1\|_{M_{2,q}^{s-1}} \leq \varepsilon$$

then, for equation (1.3), there exists a unique global

$$u \in l_{\square}^{s,q}(L^\infty(R; L^p(\mathbb{R}^n))) \cap l_{\square}^{s-\frac{\alpha}{2},q}(L^4(R; L^p(\mathbb{R}^n))). \quad (1.12)$$

Remark 5. *In [12], Miao and Zhang studied equation (1.11) on the Besov spaces. In the case $n = 3$, they showed that the exponent μ must satisfy*

$$\mu = \frac{6(s+1)}{3+\eta}$$

and

$$\mu \geq \frac{6}{2+\eta}$$

where $\eta \in [0, 1]$. These requirements imply $s \geq \frac{1}{2+\eta}$. So the minimum regularity should be $s = \frac{1}{3}$ when we choose $\eta = 1$. But on the modulation space in Theorem 1.9, if we let $\theta = 1$, we can choose p such that $2n(1 - \frac{2}{p}) = \frac{3}{2}$ when $n = 3$. Then, if we choose q closed to 1, it is not difficult to find that the minimum value of s can

be 0. We observe that in [22], Wang and Hudzik proved that $M_{p,q}$ has no derivative regularity for any $0 < p, q \leq \infty$. Hence, our result is another form of low regularity for global solution of NLKGH. Second, If one wants to obtain some high regularity estimate for this equation, in [12] the domain of μ is $\frac{3}{2}(s+1) \leq \mu \leq 2(s+1)$ when the authors take the Besov space as the working space. Checking the domain of μ on modulation spaces in Theorem 4, clearly it is larger, since the low bound is fixed which is independent on s . So, both in low regularity and high regularity cases, modulation spaces seems better than Besov spaces.

We are not surprising that the modulation space have these advantages comparing to the Besov space. In the Besov spaces, many estimates, such as the admissible pairs, Hölder's inequality, boundedness of fractional integral operator, the Sobolev embedding, etc., rely all on the exponent p , while the index q is dummy. But in the modulation spaces, the admissible pair relies on p , the Sobolev embedding relies on q (see (2.2)), Hölder's inequality and boundedness of fractional integral operator rely on both p and q . Moreover, we have the uniformly estimate for the index p (see (2.1)). In other words, working in the Besov spaces, one needs to give too much restrictions of p , and q plays no role. In the modulation spaces, p and q share these restrictions together, and both p and q have uniformly estimates.

The proofs of theorems will be represented in the third section.

2. PRELIMINARIES

In this section we recall the definitions and some properties of the modulation space and Besov space. Also, we will prove several lemmas, particularly a key lemma to estimate $|u|^k u$ in the modulation space when k is not an integer.

Definition 1. (*Modulation spaces*) Let $\{\varphi_k\} \subset C_0^\infty(\mathbb{R}^n)$ be a partition of the unity satisfying the following conditions:

$$\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \sqrt{n}\}, \quad \sum_{k \in \mathbb{Z}^n} \varphi_k(\xi) = 1,$$

for any $\xi \in \mathbb{R}^n$, where $\varphi_k(\xi) := \varphi(\xi - k)$. For each $k \in \mathbb{Z}^n$, denote a local square projection \square_k on the frequency space by

$$\square_k := \mathcal{F}^{-1} \varphi_k \mathcal{F},$$

where \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse, respectively. By this frequency-uniform decomposition operator, we define two kinds of modulation spaces, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, by

$$M_{p,q}^s(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f\|_{M_{p,q}^s(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_p^q \right)^{\frac{1}{q}} < \infty\}$$

and

$$l_{\square}^{s,q}(L^r(0,T;L^p(\mathbb{R}^n))) := \{f(t, \cdot) \in S'(\mathbb{R}^n) : \|f\|_{l_{\square}^{s,q}(L^r(0,T;L^p(\mathbb{R}^n)))} < \infty\},$$

where

$$\|f\|_{l_{\square}^{s,q}(L^r(0,T;L^p(\mathbb{R}^n)))} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_{L^r(0,T;L^p(\mathbb{R}^n))}^q \right)^{\frac{1}{q}}$$

and $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$ (See [22] for details). If the domain of t is $(-\infty, +\infty)$, we denote $l_{\square}^{s,q}(L^r L^p)$ for convenience. The space $l_{\square}^{s,q}(L^r(0,T;L^p(\mathbb{R}^n)))$ was first introduced by Planchon [14],[15] when he studied the nonlinear Schrödinger equation

and the nonlinear wave equation. In the definition, the order of L^r norm and l^q norm is changed. This change seems important in modulation spaces. As we know, q is a very important index in modulation spaces which can impact the regularity. So, in many cases, we should deal with q carefully and choose l^q norm in the last step. Moreover, we will recall some properties of modulation spaces which will be useful in this paper. More details can be found in [22].

In the following content, if no special explanation, we always assume that

$$s, s_i \in \mathbb{R}, 1 \leq p, p_i, q, q_i \leq \infty.$$

Proposition 1. (Isomorphism [22]).

Let $0 < p, q \leq \infty, s, \sigma \in \mathbb{R}$. For the Bessel potential $J_\sigma = (I - \Delta)^{\frac{\sigma}{2}}$, the mappings

$$J_\sigma : M_{p,q}^s \rightarrow M_{p,q}^{s-\sigma}$$

and

$$J_\sigma : l_{\square}^{s,q}(L^r(0,T;L^p(\mathbb{R}^n))) \rightarrow l_{\square}^{s-\sigma,q}(L^r(0,T;L^p(\mathbb{R}^n)))$$

are isomorphic mappings.

Proposition 2. (Embedding, [22]).

$M_{p_1,q_1}^{s_1} \subset M_{p_2,q_2}^{s_2}$ and $l_{\square}^{s_1,q_1}(L^r(0,T;L^{p_1})) \subset l_{\square}^{s_2,q_2}(L^r(0,T;L^{p_2}))$ if

$$(i) \quad s_1 \geq s_2, \quad 0 < p \leq p_2, \quad 0 < q_1 \leq q_2 \quad (2.1)$$

$$(ii) \quad q_1 > q_2, \quad s_1 > s_2, \quad s_1 - s_2 > n/q_2 - n/q_1 \quad (2.2)$$

In (2.1), we can see that both p and q have uniform estimates, and from (2.2) we can find that the condition on q is similar to the Sobolev embedding. Now we need the relationship between modulation spaces and Besov spaces.

Lemma 1. (Embedding with Besov spaces, [22])

Assume $B_{p,q}^s$ is the Besov spaces, and $1 \leq p, q \leq \infty, s \in \mathbb{R}$. We have the following embedding:

$$M_{p,q}^{s+\sigma(p,q)} \subset B_{p,q}^s, \quad \sigma(p,q) = \max(0, n(\frac{1}{p \wedge p'} - \frac{1}{q})) \quad (2.3)$$

$$B_{p,q}^{s+\tau(p,q)} \subset M_{p,q}^s, \quad \tau(p,q) = \max(0, n(\frac{1}{q} - \frac{1}{p \vee p'})). \quad (2.4)$$

Moreover, since we will embed the modulation spaces into the Besov spaces, we need employ some nonlinear estimates in Besov spaces, particularly the estimate on the nonlinear term u^k . Recall that such estimate in Besov spaces $B_{p,q}^s$ has a long history. Cazenave obtained the case $0 < s < 1$ in [2], and Cazenave and Weissler obtained the case $1 < s < \frac{N}{2}$ in [3]. Later, Wang proved a general case in [19]. Our proof will be based on Wang's result. Since all results on the Besov space $B_{p,q}^s$ are stated for the case $q = 2$, we need the following embedding to obtain information for all $1 \leq q < \infty$, as we will handle all q in the space $M_{p,q}^s$.

Proposition 3. ([18]) Let $\epsilon > 0$, for any $1 \leq p, q_1, q_2 \leq \infty$, then we have

$$B_{p,q_1}^{s+\epsilon} \subset B_{p,q_2}^s \quad (2.5)$$

Lemma 2. (Nonlinear estimate in Besov space)

Suppose $2 \leq p < \infty, 1 \leq q \leq \infty$ and $0 \leq \delta < s < s_1 < \infty, [s - \delta] \leq k$. If they satisfy

$$k\left(\frac{1}{p} - \frac{s}{n}\right) + \frac{1}{p} - \frac{\delta}{n} = \frac{1}{p'}, \quad \frac{1}{p} - \frac{s}{n} > 0 \quad (2.6)$$

then we have

$$\| |u|^k u \|_{B_{p',q}^{s-\delta}} \preceq \| u \|_{B_{p,2}^{s_1}}^{k+1} \quad (2.7)$$

Proof: When $q = 2$, in [20] we can find the following inequality in the given condition:

$$\| |u|^k u \|_{B_{p',2}^{s-\delta}} \preceq \| u \|_{B_{p,2}^{s_1}}^{k+1}. \quad (2.8)$$

Since the domain of s is an open set, we may choose s_ϵ and δ_ϵ so that $s < s_\epsilon < s_1$ and $\delta_\epsilon < \delta$, and require them satisfy (2.5) and (2.6). So (2.7) gives the inequality

$$\| |u|^k u \|_{B_{p',2}^{s_\epsilon - \delta_\epsilon}} \preceq \| u \|_{B_{p,2}^{s_\epsilon}}^{k+1}.$$

Finally, by Proposition 3, we obtain the desired estimate

$$\| |u|^k u \|_{B_{p',q}^{s-\delta}} \preceq \| |u|^k u \|_{B_{p',2}^{s_\epsilon - \delta_\epsilon}} \preceq \| u \|_{B_{p,2}^{s_\epsilon}}^{k+1} \preceq \| u \|_{B_{p,q}^{s_1}}^{k+1}.$$

Now, with Lemma 1 we can embed the modulation space into the Besov space and invoke Lemma 2 to obtain the nonlinear estimates on the Besov spaces. Then use Proposition 3 to transfer the estimate back to the modulation spaces. This is the following lemma, which is crucial in this paper.

Lemma 3. (Nonlinear estimate in modulation spaces)

Let $1 \leq q < \infty, 2 \leq p < \infty, s \in \mathbb{R}$ $[s - r] \leq k$. Assume that $q \in [p', p], (1 - \frac{2}{p})n < r$, and

$$\max\{r - n(\frac{1}{q} - \frac{1}{p}), \frac{n}{q'} - \frac{r}{k}\} < s < \frac{n}{q'} - \frac{1}{k}(1 - \frac{2}{p})n. \quad (2.9)$$

Then we have

$$\| u^{k+1} \|_{M_{p',q}^{s-r}} \preceq \| u \|_{M_{p,q}^s}^{k+1}. \quad (2.10)$$

Proof: By Lemma 1 we have

$$\| u^{k+1} \|_{M_{p',q}^{s-r}} \preceq \| u^{k+1} \|_{B_{p',q}^{s-r+\tau(p',q)}}. \quad (2.11)$$

Since $r - n(\frac{1}{q} - \frac{1}{p}) < s$, we have $s - r + \tau(p', q) > 0$. Using Lemma 2, we obtain

$$\| u^{k+1} \|_{B_{p',q}^{s-r+(\frac{1}{q}-\frac{1}{p})n}} \preceq \| u \|_{B_{p,q}^{s-(\frac{1}{p'}-\frac{1}{q})n}}^{k+1}. \quad (2.12)$$

Choose $s_1 = s + \varepsilon$ in (2.6), then s satisfies

$$k\left(\frac{1}{p} - \frac{1}{n}\left(s - \left(\frac{1}{p'} - \frac{1}{q}\right)n\right)\right) + \frac{1}{p} - \frac{1}{n}\left(r - n\left(1 - \frac{2}{p}\right) - \varepsilon\right) = \frac{1}{p'}. \quad (2.13)$$

By (2.13) we have

$$s = \frac{n}{q'} - \frac{r}{k} + \frac{\varepsilon}{k} \quad (2.14)$$

because $\tau(p', q) + \sigma(p, q) = n(1 - \frac{2}{p}) < r$, where it is easy to find $0 < \varepsilon < r - n(1 - \frac{2}{p})$. Combining this with (2.14) and the condition of Lemma 2, we easily see that the domain of s is

$$\max\{r - n(\frac{1}{q} - \frac{1}{p}), \frac{n}{q'} - \frac{r}{k}\} < s < \frac{n}{q'} - \frac{n}{k}(1 - \frac{2}{p}).$$

Finally, we use Lemma 1 again to obtain

$$\|u^{k+1}\|_{B_{p',q}^{s-r+(\frac{1}{q}-\frac{1}{p})n}} \preceq \|u\|_{B_{p,q}^{s-(\frac{1}{p'}-\frac{1}{q})n}}^{k+1} \preceq \|u\|_{M_{p,q}^s}^{k+1}.$$

The lemma is proved.

Remark 6. *The condition $q \in [p', p]$ is not necessary, but only for continence in the calculation. So it does not mean that q must be equal to 2 when $p = 2$. In fact, we can find a larger domain of q when $p = 2$. More precisely, with the same method as above, we may obtain the estimate*

$$\|u^{k+1}\|_{M_{2,q}^{s-r}} \preceq \|u\|_{M_{2,q}^s}^{k+1},$$

for $q < 2$ and

$$\max\{r + \frac{n}{2} - \frac{n}{q}, \frac{n}{2} - \frac{1}{k}[r - (\frac{1}{q} - \frac{1}{2})n]\} < s < \frac{n}{2}, \quad n(\frac{1}{q} - \frac{1}{2}) < r \quad (2.15)$$

or for $q > 2$ and

$$\max\{r, \frac{n}{q'} - \frac{1}{k}[r - n(\frac{1}{2} - \frac{1}{q})]\} < s < \frac{n}{q'}, \quad n(\frac{1}{2} - \frac{1}{q}) < r. \quad (2.16)$$

Remark 7. *For $1 \leq p < 2$, if we switch p and p' in the condition of Lemma 3, the similar conclusion will be obtained, that is*

$$\| |u|^k u \|_{B_{p,q}^{s-\delta}} \preceq \|u\|_{B_{p',q}^{s_1}}^{k+1}. \quad (2.17)$$

Note that Lemma 3 addresses only the estimate for $k \notin \mathbb{Z}$. When $k \in \mathbb{Z}^+$, we can find the following result in [9], which will be useful in the proof of Theorem 4

Lemma 4. *Let $s \geq 0, 1 \leq p, q, p_i, q_i \leq \infty (i = 1, 2, 3, 4)$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} + 1 = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}, \quad (2.18)$$

We have

$$\|uv\|_{M_{p,q}^s} \preceq \|u\|_{M_{p_1,q_1}^s} \|v\|_{M_{p_2,q_2}} + \|u\|_{M_{p_3,q_3}} \|v\|_{M_{p_4,q_4}^s} \quad (2.19)$$

This conclusion also holds for $l_{\square}^{s,q}(L^r L^p)$. That is, for $1 \leq r, r_i \leq \infty (i = 1, 2, 3, 4)$ satisfying

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4},$$

we have

$$\|uv\|_{l_{\square}^{s,q}(L^r L^p)} \preceq \|u\|_{l_{\square}^{s,q_1}(L^{r_1} L^{p_1})} \|v\|_{l_{\square}^{0,q_2}(L^{r_2} L^{p_2})} + \|u\|_{l_{\square}^{0,q_3}(L^{r_3} L^{p_3})} \|v\|_{l_{\square}^{s,q_4}(L^{r_4} L^{p_4})} \quad (2.20)$$

The second task of this paper is to find the global solution of equation (1.11) with small initial value. As we all know, the crucial part of the proof is the estimate of nonlinear part. To this end, we also need the estimate of fractional integral operator in modulation spaces. Recall that the fractional integral operator is define by

$$I_{\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy.$$

Lemma 5. (*Boundedness of fractional integral operator in modulation space*) [17]
 Let $0 < \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. The fractional integral operator I_α is bounded from $M_{p_1, q_1}^s(\mathbb{R}^n)$ to $M_{p_2, q_2}^s(\mathbb{R}^n)$ or from $l_{\square}^{s, q_1}(L^r L^{p_1})$ to $l_{\square}^{s, q_2}(L^r L^{p_2})$ if and only if

$$\frac{1}{p_1} \leq \frac{1}{p_2} - \frac{\alpha}{n} \text{ and } \frac{1}{q_1} \leq \frac{1}{q_2} + \frac{\alpha}{n}. \quad (2.21)$$

3. PROOF OF THE MAIN THEOREMS

Before we present the proofs, we need to state the Strichartz estimates of NLKG in modulation spaces. This estimate on the modulation spaces was obtained in [22].

Lemma 6. (*Strichartz estimate of NLKG in modulation spaces* [22]).

Let $2 \leq p < \infty, 1 \leq q < \infty, \gamma \geq 2 \vee (2/\delta)$, where α and δ are defined in (1.9). We have following the estimates:

$$\|K'(t)f\|_{M_{p, q}^{-\alpha}} \preceq (1+t)^{-\delta} \|f\|_{M_{p', q}}, \quad (3.1)$$

$$\|K'(t)f\|_{l_{\square}^{-\alpha/2, q}(L^\gamma(R, L^p))} \preceq \|f\|_{M_{2, q}}, \quad (3.2)$$

$$\left\| \int_0^t K(t-\tau) f d\tau \right\|_{l_{\square}^{-\alpha/2, q}(L^\gamma(R, L^p))} \preceq \|f\|_{l_{\square}^{-1, q}(L^1(R, L^2))}, \quad (3.3)$$

$$\left\| \int_0^t K(t-\tau) f d\tau \right\|_{l_{\square}^{-\alpha/2, q}(L^\gamma(R, L^p))} \preceq \|f\|_{l_{\square}^{\alpha/2-1, q}(L^{\gamma'}(R, L^{p'}))}. \quad (3.4)$$

In addition, if $q \in [\gamma, \gamma']$, then we have

$$\|K'(t)f\|_{L^\gamma(R, M_{p, q}^{-\alpha/2})} \preceq \|f\|_{M_{2, q}}, \quad (3.5)$$

$$\left\| \int_0^t K(t-\tau) f d\tau \right\|_{L^\gamma(R, M_{p, q}^{-\alpha/2})} \preceq \|f\|_{L^1(R, M_{2, q}^{-1})}, \quad (3.6)$$

$$\left\| \int_0^t K(t-\tau) f d\tau \right\|_{L^\gamma(R, M_{p, q}^{-\alpha/2})} \preceq \|f\|_{L^{\gamma'}(R, M_{p, q}^{\alpha/2-1})}, \quad (3.7)$$

We also need the following boundedness of K and K' on the modulation spaces.

Lemma 7. [1]. Let $1 \leq p, q < \infty, s \in \mathbb{R}$. We have the following inequalities:

$$\|K'(t)f\|_{M_{p, q}^s} \preceq (1+t)^{n|\frac{1}{2}-\frac{1}{p}|} \|f\|_{M_{p, q}^s}, \quad (3.8)$$

$$\|K(t)f\|_{M_{p, q}^s} \preceq (1+t)^{n|\frac{1}{2}-\frac{1}{p}|} \|f\|_{M_{p, q}^{s-1}}, \quad (3.9)$$

where

$$K(t) = \frac{\sin t(I - \Delta)^{\frac{1}{2}}}{(I - \Delta)^{\frac{1}{2}}}, \quad K'(t) = \cos(I - \Delta)^{\frac{1}{2}}.$$

Proof of Theorem 1. Let δ, α and θ be defined in (1.9). Consider the mapping

$$\Phi : u \rightarrow K'(t)u_0 + K(t)u_1 - \int_0^t K(t-\tau)|u|^k u d\tau$$

on the Banach space

$$X_1 = L^\infty(0, T; M_{2, q}^s) \cap L^\rho(0, T; M_{2, q}^{s-\beta}) \quad (3.10)$$

where $\rho = \frac{2}{\delta}$ and $\beta = \frac{\alpha}{2}$. For all $2 < p \leq \infty$, we can choose θ such that $\delta(p) < 1$. So by Lemma 6, we have

$$\|\Phi(u)\|_{X_1} \leq \|u_1\|_{M_{p,q}^s} + \|u_0\|_{M_{p,q}^{s-1}} + \| |u|^k u \|_{L^1(0,T;M_{2,q}^{s-1})}. \quad (3.11)$$

Choosing $r = 1$ in Remark 6 and using the Hölder inequality, we obtain that the nonlinear term

$$\| |u|^k u \|_{L^1(0,T;M_{2,q}^{s-1})} \leq \|u\|_{L^{k+1}(0,T;M_{2,q}^s)}^{k+1} \leq T \|u\|_{L^\infty(0,T;M_{2,q}^s)}^{k+1}. \quad (3.12)$$

Combining (3.11) and (3.12), we have that

$$\|\Phi(u)\|_{X_1} \leq \|u_1\|_{M_{p,q}^s} + \|u_0\|_{M_{p,q}^{s-1}} + T \| |u|^k u \|_{X_1}. \quad (3.13)$$

On the other hand, in a similar estimate it is easy to check

$$\|\Phi(u) - \Phi(v)\|_{X_1} \leq T \left(\|u\|_{X_1}^k + \|v\|_{X_1}^k \right) \|u - v\|_{X_1}$$

Denote

$$B_M = \{u \in X_1 : \|u\|_{X_1} \leq M\}.$$

We choose $M, T > 0$ such that

$$\Phi : B_M \rightarrow B_M$$

is an onto mapping and

$$\|\Phi(u) - \Phi(v)\|_{X_1} \leq \frac{1}{2} \|u - v\|_{X_1}.$$

Thus, we complete the proof of theorem by the standard method of contraction mapping.

In the proofs of Corollary 1 and Theorem 2, we can not use the Stricharz estimate in the space

$$X_2 = L^\gamma(0, T; M_{p,q}^s).$$

Hence we will invoke the boundedness of Klein-Gordon semigroup in modulation spaces to estimate the linear part.

In the proof of corollary 1, we first choose $\theta = 0$ in (1.9) such that $\alpha = \delta = 0$ in (3.1). So we obtain $\|K(t)f\|_{M_{p,q}^s} \leq \|f\|_{M_{p',q}^s}$. Now by Lemma 7, we obtain

$$\|\Phi(u)\|_{X_2} \leq (1+T)^{n|\frac{1}{2}-\frac{1}{p}|} (\|u_0\|_{M_{p,q}^s} + \|u_1\|_{M_{p,q}^{s-1}}) + \left\| \int_0^t |u|^k u d\tau \right\|_{L^\gamma(0,T;M_{p',q}^{s-1})}.$$

Choosing $r = 1$ in Lemma 3 and using Hölder's inequality, we have

$$\|\Phi(u)\|_{X_2} \leq (1+T)^{n|\frac{1}{2}-\frac{1}{p}|} (\|u_0\|_{M_{p,q}^s} + \|u_1\|_{M_{p,q}^{s-1}}) + T^{1-\frac{k}{\gamma}} \|u\|_{X_2}^{k+1}. \quad (3.14)$$

The rest of the proof is the same as that of Theorem 1 with the help of the method of contraction mapping.

To prove Theorem 2, we use Lemma 7 again to obtain

$$\begin{aligned} \|\Phi(u)\|_{X_2} &\leq \|1 + t^{(\frac{1}{2}-\frac{1}{p})n}\|_{L^\gamma(0,T)} (\|u_1\|_{M_{p,q}^s} + \|u_0\|_{M_{p,q}^{s-1}}) + \left\| \int_0^t K(t-\tau) |u|^k u d\tau \right\|_{X_2} \\ &\leq (T^{\frac{1}{\gamma}} + T^{(\frac{1}{2}-\frac{1}{p})n+\frac{1}{\gamma}}) (\|u_1\|_{M_{p,q}^s} + \|u_0\|_{M_{p,q}^{s-1}}) + \left\| \int_0^t K(t-\tau) |u|^k u d\tau \right\|_{X_2}. \end{aligned}$$

By choosing $r = 1$ in Remark 6 and Remark 7, we use Hölder's inequality to obtain the following estimate for the nonlinear term:

$$\begin{aligned}
\left\| \int_0^t K(t-\tau) |u|^k u d\tau \right\|_{X_2} &\leq \left\| \int_0^t [1 + (t-\tau)]^{n(\frac{1}{p}-\frac{1}{2})} \|u^{k+1}\|_{M_{p,q}^{s-1}} d\tau \right\|_{L^\gamma(0,T)} \\
&\leq \left\| \int_0^t [1 + (t-\tau)]^{n(\frac{1}{p}-\frac{1}{2})} \|u\|_{M_{p,q}^s}^{k+1} d\tau \right\|_{L^\gamma(0,T)} \\
&\leq \|u\|_{L^\gamma(0,T;M_{p,q}^s)}^{k+1} \cdot \|t^{\frac{\gamma-k-1}{\gamma}} (1+t^{1+n(\frac{1}{p}-\frac{1}{2})})\|_{L^\gamma(0,T)} \\
&\leq (1+T^{1+n(\frac{1}{p}-\frac{1}{2})}) T^{\frac{\gamma-k}{\gamma}} \|u\|_{X_2}^{k+1}.
\end{aligned}$$

If we first assume $T < 1$, from the above estimates we obtain that

$$\|\Phi(u)\|_{X_2} \leq C_T [\|u_1\|_{M_{p,q}^s} + \|u_0\|_{M_{p,q}^{s-1}}] + \|u\|_{X_2}^{k+1}. \quad (3.15)$$

Also, a similar method gives

$$\|\Phi(u) - \Phi(v)\|_{X_2} \leq C_T (\|u\|_{X_2}^k + \|v\|_{X_2}^k) \|u - v\|_{X_2}, \quad (3.16)$$

where the constant $C_T \rightarrow 0$, as $T \rightarrow 0$. Now, by (3.15) and (3.16), the contraction mapping yields the conclusion of Theorem 2.

Proof of Theorem 3. We denote the space

$$X = L^\infty(R; M_{2,q}^s) \cap L^{k+2}(R; M_{p,q}^{s-\frac{\alpha}{2}}),$$

where α and δ are defined in (1.9), and $k+2 \geq 2 \vee (\frac{2}{\delta})$. Since $\delta < 1$, $q \in [p', p]$, we can choose q such that $(k+2)' \leq q \leq \frac{2}{\delta}$. So we may assume $k+2 \geq \frac{2}{\delta}$ for convenience. By Lemma 6, we have

$$\|\Phi(u)\|_X \leq \|u_0\|_{M_{2,q}^s} + \|u_1\|_{M_{2,q}^{s-1}} + \left\| \int_0^t K(t-\tau) |u|^k u d\tau \right\|_{L^{k+2}(R; M_{p,q}^{s-\frac{\alpha}{2}})}.$$

The last term above can be estimated in the following by using Lemma 3 and choosing $r = 1 - \alpha$, $s = s - \frac{\alpha}{2}$ in the lemma. An easy computation gives

$$\left\| \int_0^t K(t-\tau) |u|^k u d\tau \right\|_{L^{k+2}(R; M_{p,q}^{s-\frac{\alpha}{2}})} \preceq \|u^{k+1}\|_{L^{\frac{k+2}{k+1}}(R; M_{p',q}^{s+\frac{\alpha}{2}-1})} \preceq \|u\|_{L^{k+2}(R; M_{p',q}^{s-\frac{\alpha}{2}})}^{k+1}.$$

So, we have

$$\|\Phi(u)\|_X \leq \|u_0\|_{M_{2,q}^s} + \|u_1\|_{M_{2,q}^{s-1}} + \|u\|_X^{k+1}.$$

By the standard method used in the proofs of Theorem 1 and Theorem 2, we can obtain the existence and uniqueness of global solution if the initial value $\|u_0\|_{M_{2,q}^s} + \|u_1\|_{M_{2,q}^{s-1}}$ is small enough.

Finally, we find the domain of k . We notice that when we use Lemma 3, the index p should satisfy

$$n(1 - \frac{2}{p}) < 1 - \alpha.$$

This gives

$$2n(\frac{1}{2} - \frac{1}{p}) < 1 - (n+1)\theta(\frac{1}{2} - \frac{1}{p})$$

by a simply calculation. So we find that the domain of p is $(\frac{1}{2} - \frac{1}{p}) > \frac{1}{2n+(n+1)\theta}$. The condition $k + 2 \geq \frac{2}{\delta}$ generates

$$k \geq \frac{2}{\delta} - 2 > \frac{4}{\theta} + \frac{2}{n}.$$

Proof of Theorem 4. Checking the above proof, we know that the global existence theory for small initial data is a straightforward result of the nonlinear estimate. Thus the main issue is to obtain an estimate for the nonlinear part. We use Y to denote the space

$$Y = l_{\square}^{s,q}(L^{\infty}(R; L^p(R^n))) \cap l_{\square}^{s-\frac{\theta}{2},q}(L^4(R; L^p(R^n))).$$

By Lemma 4, we obtain the following estimate for the nonlinear part:

$$\begin{aligned} \|(|x|^{-\mu} * |u|^2)u\|_{l_{\square}^{s+\frac{\theta}{2}-1,q}(L_T^{4/3}L^{p'})} &\preceq \| |x|^{-\mu} * |u|^2 \|_{l_{\square}^{s+\frac{\theta}{2}-1,q_1}(L_T^2L^{p_1})} \|u\|_{l_{\square}^{0,q_2}(L_T^4L^{p_2})} \\ &\quad + \| |x|^{-\mu} * |u|^2 \|_{l_{\square}^{0,q_3}(L_T^4L^{p_3})} \|u\|_{l_{\square}^{s+\frac{\theta}{2}-1,q_4}(L_T^2L^{p_4})} = I + II. \end{aligned}$$

We will only estimate term I , since the second term II can be estimated in the same way. Using Proposition 2, Lemma 4 and Lemma 5, we have

$$\begin{aligned} I &\preceq \|u^2\|_{l_{\square}^{s+\frac{\theta}{2}-1,q_5}(L_T^2L^{p_5})} \|u\|_{l_{\square}^{0,q_2}(L_T^4L^{p_2})} \\ &\preceq \|u\|_{l_{\square}^{s+\frac{\theta}{2}-1,q_6}(L_T^4L^{p_6})} \|u\|_{l_{\square}^{0,q_7}(L_T^4L^{p_7})} \|u\|_{l_{\square}^{0,q_2}(L_T^4L^{p_2})} \\ &\preceq \|u\|_{l_{\square}^{s-\frac{\theta}{2},q}(L_T^4L^p)}^3. \end{aligned}$$

Similarly, we can obtain

$$II \preceq \|u\|_{l_{\square}^{s-\frac{\theta}{2},q}(L_T^4L^p)}^3.$$

Therefore, we have

$$\|(|x|^{-\mu} * |u|^2)u\|_{l_{\square}^{s+\frac{\theta}{2}-1,q}(L_T^{4/3}L^{p'})} \preceq \|u\|_{l_{\square}^{s-\frac{\theta}{2},q}(L_T^4L^p)}^3 \preceq \|u\|_Y^3.$$

Again, by the standard method of contraction mapping, we prove the conclusion of the theorem.

Finally we check the range of μ . In the above proof, we notice that the conditions in the estimate of I (the same condition in the estimate of II) imply that p should satisfy

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} &= \frac{1}{p'}, \\ \frac{1}{p_6} + \frac{1}{p_7} &= \frac{1}{p_5}, \\ \frac{1}{p_1} &\leq \frac{1}{p_5} - \frac{n-\mu}{n}, \end{aligned}$$

and

$$p_2, p_6, p_7 \leq p.$$

These clearly yield that

$$2n(1 - \frac{2}{p}) \leq \mu.$$

Also, we note that q should satisfy

$$\begin{aligned}\frac{1}{q_1} + \frac{1}{q_2} &= \frac{1}{q} + 1, \\ \frac{1}{q_6} + \frac{1}{q_7} &= \frac{1}{q_5} + 1, \\ \frac{1}{q_1} &\leq \frac{1}{q_5} + \frac{n-\mu}{n}, \\ \frac{n}{q_7}, \frac{n}{q_2} &< s - \frac{\alpha}{2} + \frac{n}{q},\end{aligned}$$

and

$$s + \frac{\alpha}{2} - 1 + \frac{n}{q_6} < s - \frac{\alpha}{2} + \frac{n}{q}.$$

Thus, a direction computation gives that

$$\mu \leq 2s + \frac{2n}{q} + 1 - 2\alpha - n.$$

So, the domain of μ is

$$2n(1 - \frac{2}{p}) \leq \mu \leq 2s + \frac{2n}{q} + 1 - 2\alpha - n.$$

To compare the low bound of μ to that in [12] when $n = 3$, we choose $\theta = 1$ in (1.9). The condition $4 \geq \frac{2}{\delta}$ means that $4(1 - \frac{2}{p}) \geq \frac{1}{2}$. Since $n = 3$, the value $2n(1 - \frac{2}{p})$ is at least $\frac{3}{4}$.

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